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1979 J. Phys. A: Math. Gen. 12 2521

(http://iopscience.iop.org/0305-4470/12/12/030)

J. Phys. A: Math. Gen., Vol. 12, No. 12, 1979. Printed in Great Britain

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Received 4 December 1978, in final form 5 April 1979

Abstract. The solution of the Dirac equation for the electron in the inhomogeneous magnetic field $H_z = H\ \mathrm{sech}^2(ay)$ is discussed and it is shown that spontaneous electron–positron pair creation is possible, provided certain relations between the strength of the field, the inhomogeneity parameter and momenta are satisfied. The realisation of this possibility in the case of a neutron star is examined.

1. Introduction

It is well known that the two solutions corresponding to positive and negative energies of the Dirac equation for the free electron are separated by a gap of $2mc^2$. One expects that the switching on of an interaction may alter the size of the gap, possibly leading to its complete closure. However, the gap persists even when the electron is subjected to weak electric fields and homogeneous magnetic fields. The spectrum of the Dirac–Coulomb equation gives an energy

$$E = [1-(Z\alpha)^2]^{1/2} mc^2 \tag{1.1}$$

in the 1s state. This is still a sizable fraction of mc^2 if the atomic number Z is $\ll \alpha^{-1}$ where $\alpha \simeq \frac{1}{137}$, the fine structure constant. The spectrum of the Dirac electron in a constant (in time) homogeneous (in space) magnetic field of arbitrary strength H along the z direction (Rabi 1928) is given by

$$E = mc^2(1+2NH/H_c + p_z/mc^2)^{1/2} \tag{1.2}$$

where

$$N = n + \tfrac{1}{2} + \tfrac{1}{2}s, \qquad n = 0, 1, 2, \ldots; \tag{1.3}$$

$s = \pm 1$, corresponding to spin up and spin down for the electron, and

$$H_c = \frac{m^2c^3}{e\hbar} = 4{\cdot}414 \times 10^{13}\ \text{gauss}. \tag{1.4}$$

H_c is the critical magnetic field beyond which electrodynamics is expected to break down (Landau and Lifshitz 1961). Obviously the energy E in equation (1.2) can never reach zero (in fact the gap widens).

There are situations where the closing of the gap leading to spontaneous creation of e^+e^- pairs has been or could be claimed.

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0305-4470/79/122521+06\$01.00

1.1. *Electron in a strong nuclear Coulomb field*

From equation (1.1) we see that for $Z \simeq 137$, E vanishes. Actually a finite extension for the potential source will push the value of Z to 170 and beyond (Pomeranchuk and Smorodinsky 1945). Though it has been claimed that the inclusion of vacuum polarisation will nullify the gap closure (Panchapakesan 1971) a redefinition of the vacuum seems to restore the effect (Brodsky and Mohr 1977).

1.2. *Electron with anomalous magnetic moment placed in a homogeneous magnetic field*

Ternov *et al* (1966) solved the Dirac equation for this case. They found for the ground state energy of the electron

$$E = mc^2|1-(\alpha/4\pi)(H/H_c)| \tag{1.5}$$

which vanishes for

$$H = (4\pi c\hbar/e^2)H_c = 7\cdot 6\times 10^{16} G, \tag{1.6}$$

giving rise to the possibility of spontaneous pair creation, as has been pointed out by O'Connell (1968) and also by Chiu and Canuto (1968a) and Chiu *et al* (1968). While the maximum field strengths attainable in natural terrestrial conditions do not exceed 10^6–10^7 gauss, it is expected that fields of order 10^{13}G or even greater can exist in gravitationally collapsed bodies such as neutron stars (Chiu and Canuto 1968a, b). Indeed even in a non-magnetic star like the sun, a field strength of the order of 10^{13}G may be achieved after local gravitational collapse. The result that fields of order 10^{12}G may exist in the neighbourhood of neutron stars is based on the observed field strengths ($\simeq 10^6$G) found in white dwarfs (Angel and Landstreet 1971) and the magnetic flux conservation law which is justified by the existence of the large electrical conductivity in neutron stars (Cameron and Canuto 1974). But at the present time there does not seem to be any evidence for magnetic fields of order 10^{16}G.

According to Jancovici (1969a, b) the extrapolation of the energy expression (1.5) to higher magnetic field values, e.g. 10^{16}G as performed by O'Connell (1968) and Chiu and Canuto (1968b), is not justified. When H is large, we should also take into account higher-order terms in H. The correct way of doing this is to find the radiative corrections to the energy by considering the exact electron propagator (Demeur 1953) in a homogeneous magnetic field. Demeur obtained an integral representation for the energy which for small values of H reduces to equation (1.1) and for large values of H approximates to the expression

$$E = mc^2 + (\alpha/4\pi)mc^2\{[\ln(2e\hbar H/m^2c^3) - C - \tfrac{3}{2}]^2 + A + \cdots\}, \tag{1.7}$$

where $C = 0\cdot 577$ (Euler's constant) and A satisfies the bounds $-6 < A < 7$. From (1.7) it is seen that, irrespective of the strength of the field, the radiative correction to E remains of the order of α and E certainly does not vanish for $H = 7\cdot 6\times 10^{16}$G.

1.3. *Electron bound in a Coulomb field (hydrogen atom) in the Friedman universe*

Nowotny (1972) has discussed the energy spectrum of the hydrogen atom placed in a gravitational field with three types of Friedman line elements corresponding to spherical and hyperbolic three-dimensional spaces and flat space–time.

Using a theorem of Weyl and Titchmarsh, Nowotny finds that whereas for the flat and hyperbolic spaces a discrete spectrum exists, the H atom has only a continuous energy spectrum in the energy range $-\infty < E < \infty$ in the static, closed three-dimensional spherical space (the Einstein universe). Thus the gap from $-mc^2$ to $+mc^2$ within which the bound states, if any, should lie has been closed due to the topology, i.e. the curvature of the space–time. It is interesting that for the free electron bound up in any one of the above-mentioned types of gravitational field the gap does not close.

The purpose of the present paper is to show that in the case of electrons in the inhomogeneous magnetic field $H_z(y) = H\,\mathrm{sech}^2(ay)$ spontaneous pair creation is possible for magnetic field strengths which may be available at least in some neutron stars provided certain relations hold between the field strength, the inhomogeneity parameter and the electron momentum.

2. Energy eigenvalues for electrons in an inhomogeneous magnetic field

The Dirac equation for an electron in inhomogeneous magnetic fields has been solved exactly for four types of fields (Stanciu, 1966, 1967, Vasudevan *et al* 1967). Of these, only two cases (solved by Stanciu) give rise to bound states. To make this paper self-contained we follow his notation and give some relevant expressions.

The Dirac equation for an electron in a magnetic field lying along the z direction, $H_z(y) = H\,\mathrm{sech}^2(ay)$, arising from the vector potential $A_x(y) = -(H/a)\tanh(ay)$, can be written in the two-component form† (Case 1954, Feynman and Gell-Mann 1958)

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\mathrm{e}^2H^2}{a^2}-\frac{2p_x\,\mathrm{e}H}{a}\tanh(ay)-\mathrm{e}H\left(\frac{\mathrm{e}H}{a^2}-s\right)[\mathrm{sech}^2(ay)]\right)\psi_s\chi_s$$
$$=(E_s^2-1-p_x^2-p_z^2)\psi_s\chi_s, \tag{2.1}$$

where H is the magnetic field strength, a is the inhomogeneity parameter, p_x and p_z are the momentum components along the x and z directions respectively, $\psi_s = \phi_s(y)\exp[-\mathrm{i}(p_x x + p_z z)]$, ϕ_s being a two-component spinor function of y alone, and χ_s = spin eigenfunctions $\binom{1}{0}$, $\binom{0}{1}$, $s = \pm 1$.

For each value of s, (2.1) is the Schrödinger equation for the Rosen and Morse potential (Resen and Morse 1932, Stanciu 1966, 1967), viz

$$(-2\mathrm{e}Hp_x/a)\tanh(ay)-\{\mathrm{e}H[(\mathrm{e}H/a^2)-s]\}\,\mathrm{sech}^2(ay).$$

Defining the quantities

$$\rho=-2(\mathrm{e}H/a^3)p_x, \qquad \gamma=(\mathrm{e}H/a^2)[(\mathrm{e}H/a^2)-s],$$
$$\epsilon=[(\mathrm{e}^2H^2/a^2)+1+p_x^2+p_z^2-E_s^2]/a^2, \tag{2.2}$$

and with the transformations

$$\eta=\tfrac{1}{2}[1+\tanh(ay)], \qquad \psi_s=\exp(a\sigma y)[\cosh(ay)]^{-\tau}F_s(y), \tag{2.3}$$

where, taking positive roots throughout,

$$\sigma=-\tfrac{1}{2}[(\epsilon+\rho)^{1/2}-(\epsilon-\rho)^{1/2}], \qquad \tau=\tfrac{1}{2}[(\epsilon+\rho)^{1/2}+(\epsilon-\rho)^{1/2}], \tag{2.4}$$
$$\sigma\tau=-\tfrac{1}{2}\rho,$$

† We use the natural units $\hbar = c = m = 1$ unless otherwise stated.

the equation for $F_s(y)$ emerges as the Gaussian form of the hypergeometric equation

$$\eta(1-\eta)F_s''(\eta)+[(\sigma+\tau+1)-2\eta(\tau+1)]F_s'(\eta)+[\gamma-\tau(\tau+1)]F_s(\eta)=0. \qquad (2.5)$$

The only entire rational solutions of this equation are given by (Courant and Hilbert 1975) the Jacobi polynomials $G_n(p, q, \eta)$ which are orthogonal in the interval (0, 1) with

$$n=(\gamma+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}-\tau, \qquad n=0,1,2,\ldots \qquad (2.6)$$

and

$$q=\sigma+\tau+1>0, \qquad (2.7)$$

$$p-q=\tau-\sigma>-1. \qquad (2.8)$$

Further, from equations (2.4) and (2.6) we see that

$$\tau=(eH-Na^2)/a^2>0 \qquad (2.9)$$

with N defined in equation (1.3). The inequalities (2.7) and (2.8) are equivalent to the constraints

$$n<(\gamma+\tfrac{1}{4})^{1/2}-(\tfrac{1}{4}-\tfrac{1}{2}\rho)^{1/2}, \qquad (2.10)$$

$$n>(\gamma+\tfrac{1}{4})^{1/2}+(\tfrac{1}{4}-\tfrac{1}{2}\rho)^{1/2}. \qquad (2.11)$$

Condition (2.9) substituted in equation (2.6) rules out the possibility contained in inequality (2.11).

In addition, from equation (2.4) we see that though (2.10) is necessary, it is not sufficient and we should have

$$\sigma+\tau=(\epsilon-\rho)^{1/2}>0, \qquad \tau-\sigma=(\epsilon+\rho)^{1/2}>0, \qquad (2.12)$$

which, combined with the results

$$\sigma\tau=-\tfrac{1}{2}\rho, \qquad \tau>0, \qquad (2.13)$$

imply that

$$\tau\geqslant|\tfrac{1}{2}\rho|^{1/2}. \qquad (2.14)$$

(2.14) substituted in (2.6) yields the inequality (Stanciu 1967)

$$n\leqslant(\gamma+\tfrac{1}{4})^{1/2}-|\tfrac{1}{2}\rho|^{1/2}-\tfrac{1}{2}. \qquad (2.15)$$

The relation (2.6) gives the energy eigenvalues for the electron:

$$E_s^{e^-}=[1+p_x^2+p_z^2+2NeH-N^2a^2-p_x^2/(1-a^2N/eH)^2]^{1/2} \qquad (2.16)$$

or equivalently

$$(E_s^{e^-})^2=[(1+p_x^2+p_z^2+2N\,eH-a^2N^2)(eH-a^2N)^2-(eH)^2p_x^2]/(eH-a^2N)^2. \quad (2.17)$$

The corresponding eigenvalues $E_s^{e^+}$ for the positron are obtained by replacing N by $N'=n-\tfrac{1}{2}s+\tfrac{1}{2}$ and s by $-s$ elsewhere in the above relations.

3. Spontaneous pair creation

We express equation (2.17) in the original units for easy visualisation of the numbers involved in the solution to be given here:

$$[E_s^{e^-}]^2 = m^2c^4\{[1+(p_x/mc)^2+(p_z/mc)^2+2N(H/H_c)-a^2N^2\lambda\!\!\!^{-2}] \times(H/H_c-a^2N\lambda\!\!\!^{-2})^2-(H/H_c)^2(p_x/mc)^2\}[(H/H_c)-a^2N\lambda\!\!\!^{-2}]^{-2} \tag{3.1}$$

where $\lambda\!\!\!^-$ is the Compton wavelength of the electron:

$$\lambda\!\!\!^- = \hbar/mc = 3{\cdot}862\times10^{-11}\text{ cm}. \tag{3.2}$$

If the energy gap is to close we must have the expressions for both E_s^{e+} and E_s^{e-} going to zero subject to the constraints given in the previous section and the corresponding ones for the positron. This means, for instance, that the numerator of the right-hand side of equation (3.1) vanishes provided the denominator is not zero. This can be achieved by

(i) choosing $N=1$: this corresponds to $n=0$ and $s=1$, that is, the spin aligned along the magnetic field. Note that if the spin is opposite to the direction of the magnetic field ($s=-1$), then $N=0$ for $n=0$ and the energy reduces to

$$E_s^{e^-} = mc^2[1+(p_z/mc)^2]^{1/2} \tag{3.3}$$

with no possibility of the gap closing.

(*ii*) allowing p_x and $eH-a^2$ to behave in exactly the same way, i.e.

$$p_x/mc \simeq k>0, \qquad H/H_c-a^2\lambda\!\!\!^{-2}\simeq k>0. \tag{3.4}$$

With this choice, which is consistent with the constraint (2.15), equation (3.1) leads to

$$1+k+k^2+(p_z/mc)^2+H/H_c-(H/H_c)^2=0 \tag{3.5}$$

with the solution

$$H/H_c=\tfrac{1}{2}[\![1+\{5+4[k+k^2+(p_z/mc)^2]\}^{1/2}]\!]. \tag{3.6}$$

If k is large enough, with $p_z=0$, we can have

$$(H/H_c)\simeq k. \tag{3.7}$$

For instance, if $k=10^3$ we get

$$(H_z)_{y=0}\simeq 4{\cdot}414\times10^{16}\text{G}. \tag{3.8}$$

Although, as mentioned in the Introduction, there is no evidence for such a high magnetic field, if we assume that the magnetic fields in a neutron star are inhomogeneous, then a value for $a\simeq10^{-5}$ can bring such a value of H_z in the interior of the star down to the surface value $\simeq10^{12}$G over a thickness of 5 km from the surface. In view of (3.7) the possibility of spontaneous pair creation will exist only deep in the neutron star. The proton fluid inside the neutron star is in a superconducting state (of type I or II) and a Meissner effect could very well expel any magnetic field present in the neutron star. But Baym and Pethick (1975) argue that because of the large electrical conductivity, the time for expulsion of magnetic fields may be greater than 10^8 years, so that superconductivity and large magnetic fields could coexist.

In view of the smallness of the anomalous magnetic moment of the electron we should expect that the incorporation of the radiative corrections should not alter the above results. The computation of the self energy of the electron in the inhomogeneous field considered, using the exact Green function, is in progress and will be reported elsewhere.

Finally it is worth pointing out that for the only other inhomogeneous magnetic field $H_c = He^{ay}$ (Stanciu 1966, 1967) for which a bound state solution is known, the gap does not close.

Acknowledgments

We would like to express our deep appreciation for the interest shown by Professor S K Srinivasan in this paper. We thank the referees for constructive and valuable suggestions.

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